Dedicated stock portfolios

Portfolio managers may have been sitting atop a sleeping giant for the past thirty-eight years — Markowitz optimization techniques are about to come into their own.

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The Markowitz portfolio model introduced in 1952 expressed the optimal relationship between portfolio volatility and expected return. While the procedure was quite capable of providing perfectly accurate answers to questions such as, "Given a population of securities, what single combination would have had the lowest volatility in monthly return over the preceding period?", the computing power of the times limited its application.

A decade later, Sharpe introduced the single-index model [1963]. In essence, Sharpe's model replaced the exact, but cumbersome, Markowitz formula for portfolio volatility with a simplified approximation that assumed all the interrelationships among security returns could be attributed to the fact that they all respond differentially to the pull of the single index.

Sharpe represented his single index by the returns to the market itself, but others (King [1966], for example) soon provided evidence that other common forces were pulling security returns as well. King at the time noted industry-related factors, but since then attention has shifted from aggregate stock market- or industry-related portfolios to more generalized economic factors. In some models the factors are real (economic variables such as unexpected changes in inflation, industrial production, and term and default premiums in bond yields — see Chen, Roll, and Ross [1986]), while in others they are portfolios, derived from factor analysis (Chen [1983]).

The factor analytic portfolios allegedly mimic the behavior of the underlying economic variables actually responsible for the correlations between security returns. Index models that initially were regarded as routes to a simplified expression for portfolio volatility eventually became widely accepted as ways to track targets such as the S&P 500.

But times have changed since 1952. There have been significant advances in computing hardware and in the algorithms to compute portfolio volatility (Von Holhenbalken [1975]). Analysts are now fully capable of obtaining Markowitz solutions for several hundred securities at a time, which makes the relative computational simplicity of index models less valuable. The latest advances in hardware and software now make it possible to construct dedicated stock portfolios based on all the information in the full covariance matrix of security returns. These portfolios are dedicated to a mission of risk management such as hedging against inflation or against unexpected changes in interest rates that may dramatically increase the present value of the liabilities of pension funds.

Just as the Markowitz model provides the reactively accurate answer to the minimum volatility question posed above, it also provides the more accurate answer to the question, "Which combination of securities would have achieved the greatest degree of tracking power in the past?" While the relative predictive power of the Markowitz and index models remains an open question (and one that obviously

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depends on the form of the specific index model tested), we shall provide evidence here that the predictive power of the Markowitz model itself is quite high.

TRACKING TARGETS WITH INDEX MODELS

Index (or factor) models assume that security returns respond to the pull of one or more factors, as in the expression:

$$r_t = a + \beta_1 I_{1,t} + \beta_2 I_{2,t} + \ldots + \beta_n I_{n,t} + \epsilon_t$$

where:

- r_t = the rate of return to a security in period t;
- a = the security's expected return in a period where all factors have zero values;
- β_1 = the beta, or security responsiveness, to the first factor;
- $I_{1,t}$ = the value for the first factor in period t (for example, I_1 might be the unexpected percentage change in industrial production);
- n = the number of factors inducing correlation between security returns; and
- ϵ_t = the idiosyncratic component of the security's return in period t (that part of the return that is 1) induced by information impacting the firm only, 2) unrelated to periodic values of the factors, and 3) uncorrelated with the periodic ϵs of all other firms).

To track a target with an index model, you must first identify the relevant factors, either through a series of regression analyses with real economic variables or through a factor analysis producing portfolios that mimic the effects of the factors. Then you must estimate the betas with respect to each factor for the relevant population of securities. With respect to real economic variables, this is accomplished by regressing each security's returns on the economic variables over a past sequence of periods. With mimicking portfolios, the betas are estimated in the factor analysis as the factor loadings. Estimates of the volatility of the ϵs are also required for each security, but these are biproducts of either statistical procedure.

Similar estimates must also be made for the target.

The tracking problem can now be stated as:

Find the set of portfolio weights that minimizes idiosyncratic volatility subject to the following constraints:

1. Factor betas for the portfolio that are equal to that of the target.
2. Bounds on the portfolio weights (for example, short-selling constraints).

The exact idiosyncratic variance of the portfolio can be found using the matrix:

$$\begin{array}{ccccc}
 & x_1 & x_2 & \cdots & x_m \\
x_1 & s_1^2 & s_{1,2} & \cdots & s_{1,m} \\
x_2 & s_{2,1} & s_2^2 & \cdots & s_{2,m} \\
\cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & \cdot & & \cdot \\
x_m & s_{m,1} & s_{m,2} & \cdots & s_m^2
\end{array}$$

where:

- x_1 = portfolio weight in security 1;
- s_1^2 = idiosyncratic variance of security 1;
- $s_{1,2}$ = idiosyncratic covariance between securities 1 and 2; and
- m = number of securities in the population.

To find the idiosyncratic variance of the portfolio, you merely find the product of each variance or covariance element within the matrix and the portfolio weights at the top of its column and to the left of its row. For example, one such product for the underlined element of the matrix will be:

$$x_1 * x_2 * s_{1,2}$$

The portfolio's exact idiosyncratic covariance is the sum of the m * m products.

If you have correctly identified the factor structure, the idiosyncratic covariances between all securities will be zero. In this case, all the numbers in the matrix, except those going down the diagonal (as indicated by the arrow), will be zeros, and their associated products will be zeros as well. Thus, you need only sum the products on the diagonal, and the formula for portfolio idiosyncratic variance simplifies to:

$$s_p^2 = (x_1 * x_1 * s_1^2) + (x_2 * x_2 * s_2^2) + \ldots + (x_m * x_m * s_m^2). \quad (1)$$

If, on the other hand, you have not correctly identified the factor structure, the off-diagonal idiosyncratic covariances will not be zeros, and Equation (1) will work only as an approximation.

We should note that it is quite possible that a factor structure may not even exist. It may be impossible fully to explain with a reasonable number of factors all the interrelationships that exist among securities in large populations.

Basically, in tracking with a factor model, you identify the best possible set of factors, and you assume that Equation (1) holds as a good approximation. If you are estimating with a past sequence of returns, however, you find only the approximate answer to the question, "Which portfolio would have best tracked the target in the past period?"

TRACKING TARGETS WITH THE MARKOWITZ BULLET

Figure 1 plots the variance of returns to portfolios of securities against their beta coefficient with respect to the target. Note that the beta has replaced the traditional position of expected return on the vertical axis of the graph. Because the beta of a portfolio with a target is (as with expected return) the weighted average of the betas of the combined securities, however, the shape of the efficient opportunity set is identical to that of the conventional Markowitz bullet — a parabola (assuming no restrictions on portfolio weights). Moreover, all the geometric properties of the conventional bullet hold for this one as well. We shall be using some of these properties in variance-beta space to identify portfolios with specific tracking characteristics.

The parabolic minimum variance set shows the portfolios with lowest volatility of return, given their target beta. Portfolio variance can be written as:

$$S_P^2 = \beta_{T,P} S_T^2 + v_P^2$$

where:

S_P^2 = variance of portfolio returns;

$\beta_{T,P}$ = portfolio beta with respect to the target; and

S_T^2 = variance of the target.

Portfolios in the minimum variance set will have the smallest residual variance (v_P^2), given their betas; thus, they have the best "fits" about their characteristic lines.

Now consider the relationship between the returns to a portfolio and those of the target, as in Figure 2. One measure of tracking power is the target beta,

FIGURE 1

MINIMUM VARIANCE SET FOR TRACKING PORTFOLIOS

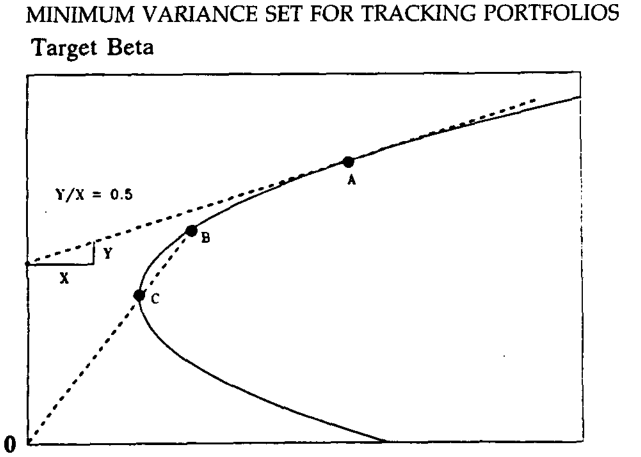
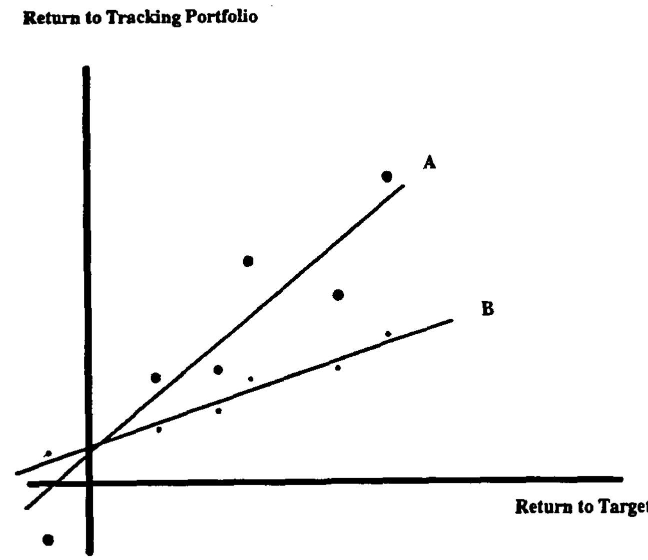


FIGURE 2

CHARACTERISTIC LINES FOR TWO TRACKING PORTFOLIOS



or the slope of the portfolio's characteristic line, relating its returns to the target's. Of the two portfolios shown in Figure 2, portfolio A has the larger target beta. As the target beta is equal to the ratio of the covariance to the variance of the target (a constant across all tracking portfolios), portfolios positioned higher on the bullet will have larger target betas.

A second measure of tracking power is the coefficient of determination (correlation squared), a measure of fit about the characteristic line, or the fraction of portfolio volatility that can be associated with target volatility. Because the correlation coefficient is equal to the ratio of 1) the product of target volatility and target beta to 2) portfolio volatility, and because target volatility is fixed, the maximum correlation portfolio will be the one maximizing the slope of a line extended from the origin in portfolio volatility-beta space, as in Figure 3. This holds regardless of restrictions placed on portfolio weights.

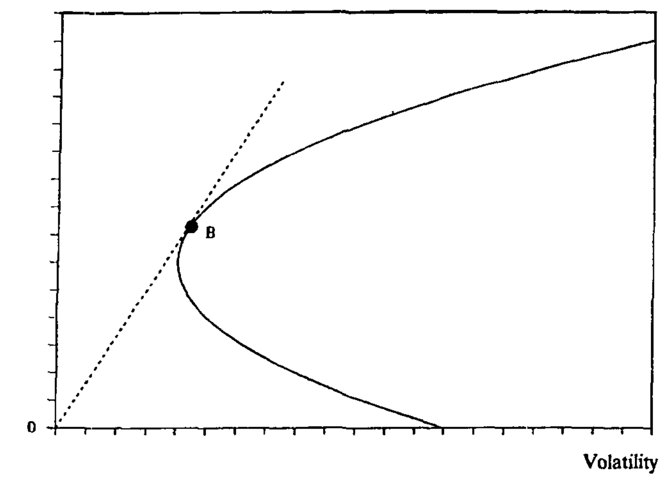
A straightforward extension of the results of Roll [1977] tells us that the maximum correlation portfolio can be found in the variance-beta space of Figure 1 by extending a line from the origin, through the global minimum variance portfolio C, to the efficient perimeter of the bullet, at portfolio B. Note that portfolios positioned to the right of B will have higher target betas but lower coefficients of determination as in Figure 2.

Given their target betas, each portfolio in the minimum variance set has the best fit about its characteristic line. Of all the portfolios in the minimum variance set, portfolio B (Figure 1 and Figure 3) has the best fit *regardless of beta.*

A third measure of tracking power is the vol-

FIGURE 3

MINIMUM VOLATILITY SET FOR TRACKING PORTFOLIOS



ability of the difference between the periodic returns to the tracking portfolio and the returns to the target. In the Appendix we show that a line (with slope equal to 0.5) that is drawn tangent to the bullet of Figure 1 identifies the unique portfolio that minimizes the volatility of return differences. Thus, while this portfolio (A) has a lower value with respect to its correlation with the target (Figure 2), its returns correspond more closely with those of the target.

In the Markowitz framework, the portfolios in the minimum variance set are the optimal tracking portfolios (ignoring assumed differences in expected return), because each minimizes volatility of residual return, given the magnitude of its beta against the target. Two of these portfolios are unique, in that one (Portfolio A in Figure 1) has the maximum coefficient of determination with the target, and the other (Portfolio B) has the minimum volatility of differences in its return from that of the target.

Similar portfolios can be found, of course, using a factor model. The best a factor approach can do, however, in finding the unique portfolios that managed to track most powerfully over a past estimation period is to equal the power of the Markowitz approach. This is possible only if the underlying factor structure has been correctly identified, and the off-diagonal elements of the matrix of idiosyncratic covariances are, thus, uniformly zero.

This is not to say that the Markowitz procedure is inherently superior in predicting the tracking prowess of alternative portfolios. Relative superiority in this regard depends on the relative stability of 1) the covariance matrix for total return, including covariances with the target, and 2) the factor betas and the covariance matrix of idiosyncratic return. Factor models are user-specific, so we leave a test of the latter proposition to the advocates of these models, while we test the former using a popular target that has proved difficult to track in the past.

TRACKING THE RATE OF INFLATION WITH THE MARKOWITZ BULLET

We shall attempt to track annual changes in the consumer price index with a diversified portfolio of equities. Our tests span the period 1976 through 1988 and use a sample of 1500 stocks from the data base of Interactive Data Corporation. To insure liquidity and to minimize the effects of estimation errors caused by non-synchronized pricing, we selected stocks that had a minimum market capitalization of $200 million in 1976. To economize on computations, we rank stocks by their covariance in the two years prior to portfolio formation, then use the 200 stocks with the highest covariance.

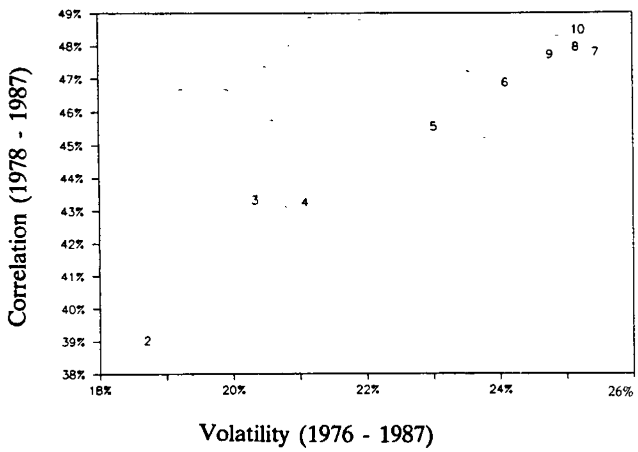
At the beginning of each year, we construct the Markowitz tracking bullet as in Figure 1, based on monthly rates of return in the preceding two years. Nine tangency portfolios from the minimum variance set are constructed with slopes of ½, ⅓, ¼, and so on through ⅒. The portfolios are bought and held for the forthcoming year. At year-end the portfolios are reformed, again based on the previous two years of monthly returns. All portfolio weights are constrained to be positive; the weights in any one stock are constrained to a maximum of 1 ⅔%; and no more than 20% of the portfolio can be invested in a given standard industrial classification (based on the first two digits).

Obviously, with no constraints on the portfolio weights, no unique solutions to our problems would exist, given the relative numbers of available securities and data points for computing the covariance matrix. With the constraints imposed on the weights, however, unique solutions do exist. These solutions, obtained by algorithms that do not require inversion of the full covariance matrix, accurately reflect the tracking performance of the portfolios they represent. To illustrate, an attempt to solve for the maximum-correlation portfolio over the previous twenty-four months of returns would, in fact, yield the unique portfolio (satisfying the constraints) having maximum correlation for the period.

An initial indication of the predictive tracking power of the Markowitz procedure appears in Figure 4, which shows the volatilities of the nine portfolios (represented by the inverse of their respective slopes relative to the ex post bullet) plotted against their correlation coefficients. Over the period, the S&P 500 had a volatility of 13.28% and a correlation with the rate of inflation of 7.1%.

FIGURE 4

VOLATILITY AND CORRELATION FOR TRACKING PORTFOLIO



Given the fact that the majority of stocks typically have negative covariance between their returns and the rate of inflation, most of the bullet (previously represented in Figure 3) lies in the southeast quadrant. (We note that the bullet is non-hyperbolic under portfolio weight constraints, making tangency solutions possible even with a negative covariance, global-minimum variance portfolio.) The southeast location of the bullet means that the maximum correlation tangency portfolios tend to be associated with relatively high levels of volatility — thus the significant positive relationship between volatility and correlation in the results.

To provide a standard for assessing the stability of the covariance matrix results, we also construct nine portfolios identical in every respect except that the bullet was constructed using returns and inflation rates for the forthcoming year (rather than the previous two years). These portfolios, constructed with perfect foresight, have an average correlation coefficient of 48.39. The similarity of the correlation levels indicates a high level of stability in the covariance matrix between the estimation and test periods.

A more extended analysis shows that the nine tracking portfolios are dominated by the influences of 1) the rate of inflation, and 2) the performance of the equity market as a whole. To show this, we regress the returns to each of the three portfolios on the rate of inflation as well as on the rate of return to the S&P 500 stock index. The results of this regression appear in the Table. The inflation betas for all nine portfolios are positive and highly significant. As the tracking procedure predicts, the betas generally increase as the targeted slope coefficient increases.

TABLE

Regression Results for

$$r_{p,t} = a + \beta_i r_{i,t} + \beta_m r_{m,t} + \epsilon_t$$

Where:

$r_{p,t}$ = rate of return on tracking portfolio in year t;

$r_{i,t}$ = rate of inflation in year t;

$r_{m,t}$ = rate of return to S&P 500 in year t.

Slope	β_i	T-Stat	β_m	T-Stat
1/2	2.261	(4.941)	1.244	(9.695)
1/3	2.678	(4.678)	1.299	(8.093)
1/4	2.771	(4.482)	1.339	(7.726)
1/5	3.156	(4.240)	1.417	(6.789)
1/6	3.386	(4.357)	1.470	(6.748)
1/7	3.633	(4.225)	1.528	(6.338)
1/8	3.601	(4.249)	1.509	(6.351)
1/9	3.531	(4.274)	1.491	(6.439)
1/10	3.640	(4.285)	1.505	(6.318)

CONCLUSION

Index models have become popular techniques for tracking targets, especially for equity portfolios. To track with an index model, you need to identify the factor structure, estimate factor betas, and then construct a portfolio with minimum idiosyncratic variance subject to the constraint that the factor betas are equal to those of the target. If the factor structure has been misspecified, portfolios will exist that have superior tracking power in the estimation period.

The Markowitz model can also be used to track targets. It can find the unique portfolio(s) that 1) maximize correlation with the target, 2) minimize volatility in the difference between their returns and the target's, or 3) minimize volatility in residual return, given their target beta. No factor structure need be estimated with the Markowitz model, and the solution portfolios have the greatest tracking power in the estimation period.

Superior tracking ability in the estimation period does not necessarily imply superior predictive power in future periods. Our test of the Markowitz model indicates nevertheless that it has remarkably high predictive power, at least in tracking annual inflation.

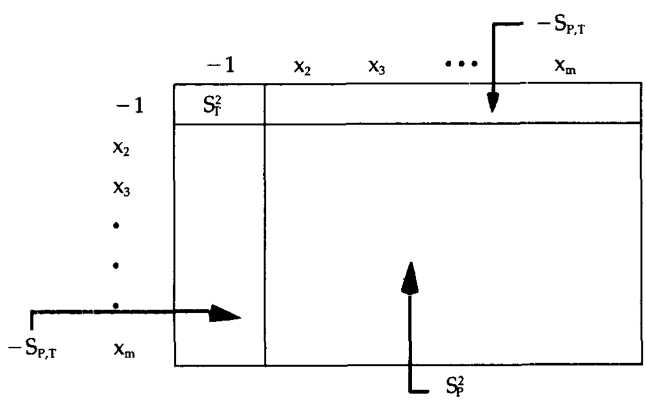
APPENDIX

To compute the volatility of the differences between the periodic returns to the portfolio and the returns to the target, you need to employ a matrix similar to the one discussed above in the section entitled "Tracking Targets with Index Models." In this case, variances and covariances of total return replace variances and covariances of idiosyncratic return. Moreover, we take the first security represented in the matrix to be the target itself. To find the minimum volatility of differences-tracking portfolio, set the weight for a target "security" equal to 100%, and find the overall minimum volatility of return portfolio. The weights in the tracking portfolio will then be $x_2, x_3, \ldots, x_m$.

The volatility of the differences for any given set of these tracking portfolio weights can be found by summing the products of all the covariance and variance elements in the matrix with all the weights at the top of the columns and the side of the rows. Dividing this sum between the tracking portfolio (with weights: x_2, x_3, . . ., x_m) and the target (with weight: x_1 = 100%) yields an expression for the variance of the return differences:

$$S_D^2 = S_P^2 \; 2(S_{P,T} + S_T^2). \tag{2}$$

To understand the expression, consider the matrix below, and note that summing the products across the top row of the matrix yields the variance of the target and the negative of the covariance between the portfolio of the target. Summing down the remaining elements of the extreme left column yields a second negative (portfolio target) covariance. All the products in the remaining elements of the matrix sum to the variance of the tracking portfolio.



Equation (2) can be manipulated to:

$$S_{P,T} = (S_T^2 \; S_D^2)\; 2 + 0.5\; (S_P^2) \tag{3}$$

Equation (3) implies that, in Figure 1, all achievable positions on a line with slope equal to 0.5 and intercept equal to $(S_D^2 + S_T^2)/2$ have identical volatility in the differences between their returns and those of the target. Thus, the line (with slope 0.5) drawn with highest possible intercept value for Equation (3) tangent to the bullet of Figure 1 is the unique portfolio that minimizes the volatility of return differences. Because Equation (2) holds under any constraints on portfolio weights, the tangency portfolio as minimum volatility portfolio holds in the general case as well.

REFERENCES

Chen, N. "Some Empirical Tests of Arbitrage Pricing." *Journal of Finance,* December 1983.

Chen, N., R. Roll, and S. Ross. "Economic Forces and the Stock Market." *Journal of Business,* July 1986.

King, B. "Market and Industry Factors on Stock Price Behavior." *Journal of Business,* January 1966.

Roll, R. "A Critique of the Asset Pricing Theory's Tests: Part I: On the Past and Potential Testability of the Theory." *Journal of Financial Economics,* March 1977.

Sharpe, W. "A Simplified Model of Portfolio Analysis." *Management Science,* January 1963.

Von Holhenbalken, B. "A Finite Algorithm to Maximize Certain Pseudo Concave Functions on Polytopes." *Mathematical Programming,* 8, 1975.